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2005 J. Phys. A: Math. Gen. 38 4865

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Scattering by local deformations of a straight leaky wire

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Received 4 October 2004, in final form 16 November 2004

Published 18 May 2005

Online at stacks.iop.org/JPhysA/38/4865

Abstract

We consider a model of a leaky quantum wire with the Hamiltonian $-\Delta - \alpha\delta(x - \Gamma)$ in $L^2(\mathbb{R}^2)$, where Γ is a compact deformation of a straight line. The existence of wave operators is proven and the S -matrix is found for the negative part of the spectrum. Moreover, we conjecture that the scattering at negative energies becomes asymptotically purely one-dimensional, being determined by the local geometry in the leading order, if Γ is a smooth curve and $\alpha \rightarrow \infty$.

PACS numbers: 02.10.Ab, 02.30.Tb, 05.45.Mt

1. Introduction

Graph models are very useful in describing a variety of mesoscopic systems: see [Ku] for a review. They have some drawbacks, however, namely that they contain free parameters in the boundary conditions describing the vertices, with no easy way to fix their values, and they neglect the quantum tunnelling between different parts of the graph. An attempt to construct models free of these deficiencies was a motivation of the recent work on *leaky quantum graphs* described formally by Schrödinger operators

$$-\Delta - \alpha\delta(x - \Gamma) \tag{1.1}$$

in $L^2(\mathbb{R}^d)$ with an attractive singular interaction supported by a graph Γ ; a precise definition of this operator, denoted as $-\Delta_\Gamma$ or $-\Delta_{\alpha,\Gamma}$, will be given below for the particular situation considered in this paper.

Various results are available concerning the discrete spectrum of such systems—see, e.g., [BT, BEKŠ], more recently [EI, EK1, EK2, EY1, EY2, EY3] and references given in these papers. On the other hand, almost nothing is known about the scattering in this context, apart

from analysis of a very simple model [EK3] and indirect indications coming from the spectral properties; both indicated that interesting resonance effects may occur [EN].

Our aim in this paper is to address this question in the simple situation when the system is planar, $d = 2$, and Γ is a local deformation of a straight line $\Sigma = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, or in other words, that the perturbation of $-\Delta_\Sigma = -\Delta - \alpha\delta(x - \Sigma)$ responsible for the scattering is a singular interaction supported by the symmetric difference of the two sets,

$$\Lambda \equiv \Gamma \Delta \Sigma := (\Gamma \setminus \Sigma) \cup (\Sigma \setminus \Gamma). \quad (1.2)$$

The spectrum of the unperturbed operator $-\Delta_\Sigma$ is easily found by separation of variables. The transverse part is reduced to the problem of a one-dimensional Laplacian with a single point interaction [AGHH]. Since the latter is attractive, it gives rise to the eigenvalue $-\alpha^2/4$ with the eigenfunction $e^{-\alpha|x|/2}$. Consequently, the two-dimensional system described by $-\Delta_\Sigma$ has a purely absolutely continuous spectrum equal to $(-\alpha^2/4, \infty)$; states with negative energies can only be transported along the line Σ .

With the singular character of the perturbation in mind, our main tool will be a Krein-type resolvent formula which we derive in section 2.2 below. The assumption about compact support will allow us to check the stability of the essential spectrum in section 2.3, and moreover, to derive the same result for the absolutely continuous spectrum and to prove the existence of the wave operators—see section 2.4. The spectral properties of $-\Delta_\Sigma$ suggest that the scattering problem looks different for positive and negative energies; in this paper we concentrate on the *negative spectrum* of $-\Delta_\Gamma$. The generalized eigenfunctions of the unperturbed operator $-\Delta_\Sigma$ corresponding to eigenvalues $\lambda \in (-\alpha^2/4, 0)$ are easily seen to be

$$\omega_\lambda(x_1, x_2) = e^{i(\lambda + \alpha^2/4)^{1/2}x_1} e^{-\alpha|x_2|/2} \quad (1.3)$$

and its complex conjugate $\bar{\omega}_\lambda$. The generalized eigenfunctions of $-\Delta_\Gamma$ will be in section 3 constructed as superpositions of ω_λ and $\bar{\omega}_\lambda$ when we are far from the scattering region Λ , so that the scattering problem is essentially one-dimensional in the sense that it is described by a 2×2 matrix of reflection and transmission amplitudes. The scattering problem for the positive part of the spectrum is more complicated and we postpone it to a subsequent paper.

The claim about one-dimensional character has to be taken *cum grano salis* because due to quantum tunnelling the scattering depends in general on the global geometry of Γ as an example worked out in [EN] suggests. One can expect, however, that such effects will be suppressed if the attractive interaction is strong enough. In the concluding remarks, we will make this claim more precise stating it as a conjecture which is expected to be valid in the asymptotic regime $\alpha \rightarrow \infty$, in the leading order at least, if Γ is a sufficiently smooth curve.

2. Scattering due to local deformation

2.1. Geometry of Γ and definition of $-\Delta_\Gamma$

Naturally we have to assume more about Γ than just its local character; we suppose that Γ is a finite family of C^1 smooth curves in \mathbb{R}^2 . We will also require that no pair of components of Γ crosses at their interior points, neither does a component have a self-intersection; we allow the components to touch at their endpoints but assume they do not form a cusp there. To summarize this survey of requirements, we assume that

- (a1) there exists a compact set $M \subset \mathbb{R}^2$ such that

$$\Gamma \setminus M = \Sigma \setminus M, \quad (2.1)$$

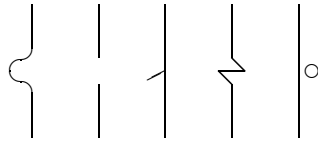


Figure 1. Examples of locally deformed lines.

(a2) the set $\Gamma \setminus \Sigma$ admits a finite decomposition,

$$\Gamma \setminus \Sigma = \bigcup_{i=1}^N \Gamma_i, \quad N < \infty, \tag{2.2}$$

where the Γ_i are finite C^1 curves with the properties described above.

Examples of such locally deformed lines are shown in figure 1. More comments on the assumptions will be given below—cf remark 2.1.

Let us next describe a proper way to define the Hamiltonian with a perturbation supported by Γ . For $i = 1, \dots, N$ we denote by ν_i the Dirac measure on Γ_i , more precisely, for a Borel set $\mathcal{B} \subset \mathbb{R}^2$ we have

$$\nu_i(\mathcal{B}) := l(\mathcal{B} \cap \Gamma_i),$$

where $l(\cdot)$ is the one-dimensional Hausdorff measure given by the arc length of Γ_i ; in a similar way we define the measure $\tilde{\nu}$ on $\Sigma \cap \Gamma$. Then the sum $\eta := \tilde{\nu} + \sum_{i=1}^N \nu_i$ is a Dirac measure on Γ and it follows from theorem 4.1 of [BEKŠ] that it belongs to the generalized Kato class; recall that a measure m in \mathbb{R}^2 is said to belong to the generalized Kato class if

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \mathbb{R}^2} \int_{B_\varepsilon(x)} |\ln|x - y|| dm(y) = 0,$$

where $B_\varepsilon(x)$ is the ball of radius ε centred at x . We also introduce the space $L^2(\eta) \equiv L^2(\mathbb{R}^2, \eta)$ which admits the direct sum decomposition

$$L^2(\eta) = L^2(\tilde{\nu}) \oplus \left(\bigoplus_{i=1}^N L^2(\nu_i) \right). \tag{2.3}$$

A rigorous definition of $-\Delta_\Gamma$ can be given in terms of the following quadratic form:

$$\gamma(f, g) := (\nabla f, \nabla g) - \alpha(I_\Gamma f, I_\Gamma g)_{L^2(\eta)} \quad \text{for } f, g \in W^{2,1} \equiv W^{2,1}(\mathbb{R}^2), \tag{2.4}$$

where (\cdot, \cdot) is the scalar product in $L^2 \equiv L^2(\mathbb{R}^2)$ and I_Γ is the standard embedding operator acting from $W^{2,1}$ to $L^2(\eta)$. For brevity we will write in the following $(f, g)_{L^2(\eta)} = (I_\Gamma f, I_\Gamma g)_{L^2(\eta)}$ assuming that the functions $f, g \in W^{2,1}$ are embedded in $L^2(\eta)$, and the same self-explanatory notation will be used for other spaces with Dirac measures. Since the measure η is of the Kato class we infer that the form (2.4) is closed, see [BEKŠ] for details, and therefore the operator associated with it is self-adjoint; we identify it with the Hamiltonian of the problem given formally by (1.1).

Remark 2.1. The assumptions can be slightly weakened. For instance, one can require only that the components Γ_i are only *piecewise* C^1 smooth, which is equivalent to gluing several of them into a single curve. Since the corresponding η has to belong to the generalized Kato class cusps must be avoided. To formulate a sufficient condition for that let us parametrize Γ_i by its arc length, i.e. regard it as a graph of the function $(0, L) \ni s \mapsto \Gamma_i(s) \in \mathbb{R}^2$. Cusps will then be absent if there are constants $C_i > 0$ such that

$$|\Gamma_i(s) - \Gamma_i(s')| \geq C_i |s - s'| \quad \text{for } s, s' \in (0, L). \tag{2.5}$$

The same condition can be used for Γ with branching points; one has to take all possible piecewise smooth curves which are subsets of such a Γ and to demand that they satisfy the above inequality.

2.2. Krein-type formula for the resolvent of $-\Delta_\Gamma$

Since $-\Delta_\Gamma$ is a singular perturbation of $-\Delta$ one can write the corresponding relation between the resolvents—cf [EI]. For our purposes, however, it is more useful to regard it as a singular perturbation of $-\Delta_\Sigma$ by a δ potential supported by the set Λ which will be decomposed as follows:

$$\Lambda = \Lambda_0 \cup \Lambda_1 \quad \text{with} \quad \Lambda_0 := \Sigma \setminus \Gamma, \quad \Lambda_1 := \Gamma \setminus \Sigma = \bigcup_{i=1}^N \Gamma_i; \tag{2.6}$$

the coupling constant of the potential representing the perturbation will be naturally equal to α on Λ_0 and $-\alpha$ on Λ_1 .

Recall first what the resolvent of $-\Delta_\Sigma$ looks like. Assume that $\text{Im } k > 0$ and k^2 belongs to the resolvent set of the free Laplacian, $k^2 \in \rho(-\Delta)$, and denote by R^k the resolvent of $-\Delta$, which is an integral operator with the kernel

$$G^k(x-y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ip(x-y)}}{p^2 - k^2} dp = \frac{1}{2\pi} K_0(ik|x-y|),$$

where $K_0(\cdot)$ stands for the Macdonald function. To construct the resolvent of $-\Delta_\Sigma$ we need the embeddings of R^k to spaces canonically associated with Σ . Let $\mu_\Sigma \equiv \mu$ be the Dirac measure on Σ ; by means of it we define the operator

$$R_\mu^k : L^2(\mu) \rightarrow L^2, \quad R_\mu^k f = G^k * f \mu$$

with the adjoint $(R_\mu^k)^* : L^2 \rightarrow L^2(\mu)$ and $R_{\mu\mu}^k$ which is the integral operator with the same kernel as R_μ^k but acting from $L^2(\mu)$ to $L^2(\mu)$. Using the natural isomorphism $L^2(\mu) \cong L^2(\mathbb{R})$ the kernel of $R_{\mu\mu}^k$ can be written as

$$G_{\mu\mu}^k(x_1 - y_1) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{e^{ip_1(x_1-y_1)}}{\tau_k(p_1)} dp_1, \tag{2.7}$$

where $\tau_k(p_1) := (p_1^2 - k^2)^{1/2}$. Given $k^2 \in \rho(-\Delta_\Sigma) = \mathbb{C} \setminus (-\alpha^2/4, \infty)$ with $\text{Im } k > 0$ we can express the resolvent of $-\Delta_\Sigma$ in the following form:

$$R_\Sigma^k = R^k + \alpha R_\mu^k (1 - \alpha R_{\mu\mu}^k)^{-1} (R_\mu^k)^*,$$

which is, of course, a particular case of a general formula given in [EI]. A straightforward calculation using (2.7) shows that R_Σ^k is an integral operator with the kernel

$$G_\Sigma^k(x-y) = G^k(x-y) + \frac{\alpha}{4\pi^3} \int_{\mathbb{R}^3} \frac{e^{ipx - ip'y}}{(p^2 - k^2)(p'^2 - k^2)} \frac{\tau_k(p_1)}{2\tau_k(p_1) - \alpha} dp dp'_2, \tag{2.8}$$

where we have denoted $p = (p_1, p_2)$ and $p' = (p_1, p'_2)$.

Now we are going to express the resolvent of $-\Delta_\Gamma$ understanding this operator as a singular perturbation of $-\Delta_\Sigma$. We want to derive a Krein-type formula using R_Σ^k and its appropriate embeddings to $L^2(\nu)$, where $\nu \equiv \nu_\Lambda$ is the Dirac measure on Λ . The latter allows for the following decomposition:

$$\nu = \nu_\Lambda = \nu_0 + \sum_{i=1}^N \nu_i, \tag{2.9}$$

where ν_0 is the Dirac measure on Λ_0 . It is also convenient to denote $\mathfrak{h} \equiv L^2(\nu)$; this space inherits from (2.9) the direct sum decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ with $\mathfrak{h}_0 \equiv L^2(\nu_0)$ and $\mathfrak{h}_1 \equiv \bigoplus_{i=1}^N L^2(\nu_i)$. In the same way as before we introduce the operator

$$R_{\Sigma, \nu}^k : \mathfrak{h} \rightarrow L^2, \quad R_{\Sigma, \nu}^k f = G_{\Sigma}^k * f \nu \quad \text{for } f \in \mathfrak{h} \tag{2.10}$$

which will be shown to be defined on the whole \mathfrak{h} for suitable values of k . Similarly, $(R_{\Sigma, \nu}^k)^* : L^2 \rightarrow \mathfrak{h}$ is its adjoint and $R_{\Sigma, \nu \nu}^k$ denotes the operator-valued matrix in \mathfrak{h} with the ‘block elements’ $G_{\Sigma, ij}^k \equiv G_{\Sigma, \nu_i \nu_j}^k : L^2(\nu_j) \rightarrow L^2(\nu_i)$ defined as the appropriated embeddings of (2.8). The following lemmas show that the above constructed operators are bounded at least for some k .

Lemma 2.2. *For any $a > 0$ there exists κ_a such that the inequality*

$$\|G^{i\kappa} * f \nu_i\|_{L^2(\nu_j)} \leq a \|f\|_{L^2(\nu_i)}, \quad f \in L^2(\nu_i),$$

holds for all $\kappa > \kappa_a$.

Proof. The argument is the same as in corollary 2.2 of [BEKŠ]. □

Lemma 2.3.

- (i) *For any $\kappa \in (\alpha/2, \infty)$ the operator $R_{\Sigma, \nu}^{i\kappa}$ is bounded.*
- (ii) *For any $\sigma > 0$ there exists κ_σ such that for $\kappa > \kappa_\sigma$ the operator $R_{\Sigma, \nu \nu}^{i\kappa}$ is bounded with the norm less than σ .*

Proof. (i) It suffices to establish the existence of $C > 0$ such that

$$\|G_{\Sigma}^k * f \nu_j\| \leq C \|f\|_{L^2(\nu_j)}, \quad f \in L^2(\nu_j), \tag{2.11}$$

holds for $j = 0, \dots, N$ and $k = i\kappa$, where $\kappa \in (\alpha/2, \infty)$. One can consider the terms on the rhs of (2.8) separately. For the first component G^k of G_{Σ}^k it follows from Sobolev embedding theorem. Let us denote the second component of G_{Σ}^k in (2.8) by ξ^k . For any $\kappa \in (\alpha/2, \infty)$ we have the inequality

$$0 < \frac{\tau_{i\kappa}(p_1)}{2\tau_{i\kappa}(p_1) - \alpha} < M_\kappa \quad \text{for } p_1 \in \mathbb{R} \tag{2.12}$$

with a constant $M_\kappa > 0$. Hence an elementary estimate gives

$$\|\xi^k * f \nu_j\|^2 \leq C_1 \|f\|_{L^1(\nu_j)}^2 \int_{\mathbb{R}^2} \frac{1}{(p^2 + \kappa^2)^2 (p_1^2 + \kappa^2)} dp \leq C_2 \|f\|_{L^2(\nu_j)}^2, \tag{2.13}$$

where C_1, C_2 are positive constants and $p = (p_1, p_2)$; in the last inequality we have used the fact that ν_j has a compact support. This yields (2.11).

(ii) Another straightforward estimate relying on (2.12) yields

$$\|\xi^k * f \nu_j\|_{L^2(\nu_j)}^2 \leq C'_1 \|f\|_{L^1(\nu_j)}^2 \left(\int_{\mathbb{R}} \frac{1}{p_1^2 + \kappa^2} \right)^2 dp_1 \leq C'_2 \kappa^{-2} \|f\|_{L^2(\nu_j)}^2,$$

for each $i, j = 0, \dots, N$, where C'_1, C'_2 are positive constants. In view of (2.12) we see that C'_1, C'_2 are in fact functions of κ but they are uniformly bounded wrt $\kappa > \kappa_0$ where $\kappa_0 \in (\alpha/2, \infty)$ is a fixed number. Combining this result with lemma 2.2 we arrive at the desired conclusion. □

Remark 2.4. Note that $R_{\Sigma, \nu}^{i\kappa}$ with $\kappa \in (\alpha/2, \infty)$ is in fact a continuous embedding into $W^{2,1}$. Indeed, the Sobolev space theory tells us that

$$\|G^k * f \nu_j\|_{W^{2,1}} < C \|f\|_{L^2(\nu_j)}, \quad f \in L^2(\nu_j).$$

On the other hand, the estimate (2.13) can be strengthened,

$$\|\xi^k * f v_i\|_{W^{2,1}}^2 \leq C_1 \|f\|_{L^1(v_i)}^2 \int_{\mathbb{R}^2} \frac{1}{(p^2 + \kappa^2)(p_1^2 + \kappa^2)} dp \leq C_2 \kappa^{-2} \|f\|_{L^2(v_i)}^2;$$

together these results give the above claim.

Let us now introduce an operator-valued matrix acting on $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ as

$$\Theta^k = -(\alpha^{-1} \check{\mathbb{I}} + R_{\Sigma, v}^k) \quad \text{with} \quad \check{\mathbb{I}} = \begin{pmatrix} \mathbb{I}_0 & 0 \\ 0 & -\mathbb{I}_1 \end{pmatrix},$$

where \mathbb{I}_i are the unit operators in \mathfrak{h}_i . By lemma 2.3 the operator $\Theta^{i\kappa}$ is boundedly invertible if κ is large enough, i.e. $(\Theta^{i\kappa})^{-1} \in \mathcal{B}(\mathfrak{h})$. Now we are ready to prove the following theorem.

Theorem 2.5. *Let $(\Theta^k)^{-1} \in \mathcal{B}(\mathfrak{h})$ hold for $k \in \mathbb{C}^+$ and let the operator*

$$R_\Gamma^k = R_\Sigma^k + R_{\Sigma, v}^k (\Theta^k)^{-1} (R_{\Sigma, v}^k)^* \tag{2.14}$$

be defined everywhere on L^2 . Then k^2 belongs to $\rho(-\Delta_\Gamma)$ and the resolvent $(-\Delta_\Gamma - k^2)^{-1}$ is given by R_Γ^k .

Proof. Note first that in view of lemma 2.3 the assertion is not empty; let us suppose for the moment that $k = i\kappa$ with κ sufficiently large. Furthermore, the quadratic form (2.4) can be rewritten as follows:

$$\gamma(f, g) = (\nabla f, \nabla g) - \alpha(f, g)_{L^2(\mu_\Sigma)} + \alpha(\check{\mathbb{I}} I_\Lambda f, I_\Lambda g)_\mathfrak{h}, \quad f, g \in W^{2,1}, \tag{2.15}$$

where I_Λ is the standard embedding of $W^{2,1}$ to $\mathfrak{h} = L^2(v_\Lambda)$. By remark 2.4 we have $f = R_\Gamma^k h \in W^{2,1}$ for $h \in L^2$, and applying (2.14) to (2.15) we get

$$\gamma(f, g) - k^2(f, g) = (h, g) + ((\Theta^k)^{-1} (R_{\Sigma, v}^k)^* h, I_\Lambda g)_\mathfrak{h} + \alpha(\check{\mathbb{I}} I_\Lambda R_\Gamma^k h, I_\Lambda g)_\mathfrak{h}. \tag{2.16}$$

To proceed further let us note that the definitions of I_Λ and Θ^k imply

$$\begin{aligned} \alpha(\check{\mathbb{I}} I_\Lambda R_\Sigma^k h, I_\Lambda g)_\mathfrak{h} &= \alpha(\check{\mathbb{I}} (R_{\Sigma, v}^k)^* h, I_\Lambda g)_\mathfrak{h} \\ &= -((\Theta^k)^{-1} (R_{\Sigma, v}^k)^* h, I_\Lambda g)_\mathfrak{h} - \alpha(\check{\mathbb{I}} R_{\Sigma, v}^k (\Theta^k)^{-1} (R_{\Sigma, v}^k)^* h, I_\Lambda g)_\mathfrak{h}. \end{aligned} \tag{2.17}$$

Applying again (2.14) to (2.16) and using (2.17) we get by a direct calculation that $\gamma(f, g) - k^2(f, g) = (h, g)$ holds for any $g \in W^{2,1}$. This is equivalent to the relation $R_\Gamma^k = (-\Delta_\Gamma - k^2)^{-1}$ for $k = i\kappa$ with κ sufficiently large, and as the resolvent of a self-adjoint operator R_Γ^k can continue analytically to the region \mathbb{C}^+ ; this completes the proof. \square

2.3. Spectrum of $-\Delta_\Gamma$

Let us turn to the description of the spectrum of our Hamiltonian. The spectrum of the unperturbed operator $-\Delta_\Sigma$ is found easily by separation of variables. The transverse part is the one-dimensional operator $-\Delta_\alpha^{(1)}$ with a single point interaction. It is well known [AGHH] that its spectrum is purely absolutely continuous in $[0, \infty)$, and in the attractive case, $\alpha > 0$, which we are interested in, there is also one eigenvalue equal to $-\frac{1}{4}\alpha^2$. Combining this with the Laplacian in the other direction, we find that the spectrum of $-\Delta_\Sigma$ is purely absolutely continuous covering the interval $[-\frac{1}{4}\alpha^2, \infty)$.

Let us first check stability of the essential spectrum.

Theorem 2.6. $\sigma_{\text{ess}}(-\Delta_\Gamma) = \sigma_{\text{ess}}(-\Delta_\Sigma) = [-\frac{1}{4}\alpha^2, \infty)$.

Proof. In view of the resolvent formula (2.14) and the Weyl theorem it is sufficient to show that there exists $k \in \mathbb{C}^+$ such that the operator

$$B^k \equiv R_{\Sigma, v}^k (\Theta^k)^{-1} (R_{\Sigma, v}^k)^*$$

is compact. It follows from lemma 2.3 that $(\Theta^{i\kappa})^{-1} \in \mathcal{B}(\mathfrak{h})$ and $(R_{\Sigma, v}^{i\kappa})^*$ is bounded if κ is large enough. Furthermore, it was shown in [BEKŠ] that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G^{i\kappa}(x-y)|^2 v_j(dy) dx < \infty. \tag{2.18}$$

On the other hand for $\kappa \in (\alpha/2, \infty)$ and $j = 0, \dots, N$ the second component of $G_{\Sigma}^{i\kappa}$ given in (2.8) can be estimated as

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\xi^k(x, y)|^2 v_j(dy) dx < CL_j \int_{\mathbb{R}^2} \frac{dp}{(p^2 + \kappa)^2} < \infty, \tag{2.19}$$

where C is a constant and L_j denote the length of Λ_j . Combining (2.18) and (2.19) we get the compactness of $R_{\Sigma, v}^k$, and thus the same for B^k . \square

Remark 2.7. Using the results of [Po] we can determine the discrete spectrum of $-\Delta_{\Gamma}$ from zeros of the operator-valued function $k \mapsto \Theta^k$. It is known, for example, that if Γ is a single non-straight curve there is at least one isolated eigenvalue below $-\frac{1}{4}\alpha^2$ [EI]. However, the discrete spectrum is not the object of our interest in this paper.

2.4. Existence of wave operators

Let us turn now to the scattering theory for the pair $(-\Delta_{\Gamma}, -\Delta_{\Sigma})$. To establish the existence of wave operators we will employ the Kuroda–Birman theorem by which it is sufficient to prove that the resolvent difference of the two operators belongs to the trace class—see [RS, section XI.3]. This is the content of the following claim.

Theorem 2.8. $B^{i\kappa}$ is a trace class operator for κ sufficiently large.

Proof. The idea is borrowed in part from [BT]. By lemma 2.3 we have

$$(\Theta^{i\kappa})^{-1} \leq C' (\Theta^{i\kappa, +})^{-1}, \quad \text{where } \Theta^{i\kappa, +} := \alpha^{-1} \mathbb{I} + R_{\Sigma, vv}^{i\kappa} \quad \text{and} \quad \mathbb{I} = \begin{pmatrix} \mathbb{I}_0 & 0 \\ 0 & \mathbb{I}_1 \end{pmatrix},$$

for some $C' > 0$ and all κ sufficiently large; it is clear that operator $(\Theta^{i\kappa, +})^{-1}$ is positive and bounded. This in turn implies the inequality

$$B^{i\kappa} \leq C' B^{i\kappa, +}, \quad B^{i\kappa, +} := R_{\Sigma, v}^{i\kappa} (\Theta^{i\kappa, +})^{-1} (R_{\Sigma, v}^{i\kappa})^*.$$

Furthermore, define $B_{\delta}^{i\kappa, +}$ as the integral operator with the kernel

$$B_{\delta}^{i\kappa, +}(x, y) = \chi_{\delta}(x) B^{i\kappa, +}(x, y) \chi_{\delta}(y),$$

where $B^{i\kappa, +}(\cdot, \cdot)$ is the kernel of $B^{i\kappa, +}$ and χ_{δ} stands for the indicator function of the ball $\mathcal{B}(0, \delta)$; one has, of course, $B_{\delta}^{i\kappa, +} \rightarrow B^{i\kappa, +}$ as $\delta \rightarrow \infty$ in the weak sense. Moreover, using the estimate from the proof of theorem 2.6 we get

$$\begin{aligned} \int_{\mathbb{R}^2} B_{\delta}^{i\kappa, +}(x, x) dx &= \int_{\mathbb{R}^2} (G_{\Sigma}^{i\kappa}(\cdot, x) \chi_{\delta}(x), (\Theta^{i\kappa, +})^{-1} G_{\Sigma}^{i\kappa}(\cdot, x) \chi_{\delta}(x))_{\mathfrak{h}} dx \\ &\leq \|(\Theta^{i\kappa, +})^{-1}\| \int_{\mathbb{R}^2} \|G_{\Sigma}^{i\kappa}(\cdot, x) \chi_{\delta}(x)\|_{\mathfrak{h}}^2 dx \leq C \|(\Theta^{i\kappa, +})^{-1}\|, \end{aligned}$$

where C is a positive constant. Next we apply the lemma following theorem XI.31 in [RS] by which the operator $B_{\delta}^{i\kappa, +}$ is trace class for any $\delta > 0$ (and κ large enough); since $\text{Tr } B_{\delta}^{i\kappa, +} \rightarrow \text{Tr } B^{i\kappa, +}$ holds as $\delta \rightarrow \infty$, the same is also true for the limiting operator. In a similar way we can construct a Hermitian trace class operator $B^{i\kappa, -}$ which provides an estimate from below, $B^{i\kappa, -} \leq B^{i\kappa}$; this means that $B^{i\kappa}$ is a trace class operator too. \square

3. Generalized eigenfunctions and the S-matrix

The existence of wave operators itself does not tell us much, we have to be able to find the S-matrix, which by definition acts as

$$S\psi_\lambda^- = \psi_\lambda^+$$

relating the incoming and outgoing asymptotic solutions. In particular, for scattering in the negative part of the spectrum with a fixed $\lambda \in (-\frac{1}{4}\alpha^2, 0)$ corresponding to the effective momentum $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$, the latter are combinations of the generalized eigenfunctions ω_λ and $\bar{\omega}_\lambda$ given by (1.3).

The functions (1.3) and their analogues ω_z for complex values of the energy parameter are L^2 only locally, of course, and we can approximate them by the family of regularized functions,

$$\omega_z^\delta(x) = e^{-\delta x_1^2} \omega_z(x) \quad \text{for } z \in \rho(-\Delta_\Sigma),$$

which naturally belong to $D(-\Delta_\Sigma)$. Consider now a function ψ_z^δ such that $(-\Delta_\Gamma - z)\psi_z^\delta = (-\Delta_\Sigma - z)\omega_z^\delta$. A direct computation gives

$$(-\Delta_\Gamma - z)\psi_z^\delta = 2\delta(2\delta x_1^2 - 1 - 2i k_\alpha(\lambda))\psi_z^\delta. \tag{3.1}$$

After taking the limit $\lim_{\epsilon \rightarrow 0} \psi_{\lambda+i\epsilon}^\delta = \psi_\lambda^\delta$ in the topology of L^2 the function ψ_λ^δ still belongs to $D(-\Delta_\Gamma)$, and moreover, we have

$$\psi_\lambda^\delta = \omega_\lambda^\delta + R_{\Sigma, \nu}^{k_\alpha(\lambda)} (\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda^\delta,$$

where $R_{\Sigma, \nu}^{k_\alpha(\lambda)}$ is the integral operator acting on the auxiliary Hilbert space \mathfrak{h} , analogous to (2.10), which is given by the kernel

$$G_\Sigma^{k_\alpha(\lambda)}(x-y) := \lim_{\epsilon \rightarrow 0} G_\Sigma^{k_\alpha(\lambda+i\epsilon)}(x-y); \tag{3.2}$$

similarly $\Theta^{k_\alpha(\lambda)} := -\alpha^{-1}\mathbb{I} - R_{\Sigma, \nu\nu}^{k_\alpha(\lambda)}$ are the operators on \mathfrak{h} with $R_{\Sigma, \nu\nu}^{k_\alpha(\lambda)}$ being the embeddings defined by means of (3.2). The explicit form of this kernel was derived in [EK2] to be⁴

$$G_\Sigma^{k_\alpha(\lambda)}(x-y) = K_0(i\sqrt{\lambda}|x-y|) + \mathcal{P} \int_0^\infty \frac{\mu_0(t; x, y)}{t - \lambda - \alpha^2/4} dt + s_\alpha(\lambda) e^{ik_\alpha(\lambda)|x_1-y_1|} e^{-\alpha/2(|x_2|+|y_2|)}, \tag{3.3}$$

where $s_\alpha(\lambda) := i\alpha(2^3 k_\alpha(\lambda))^{-1}$ and

$$\mu_0(t; x, y) := -\frac{i\alpha}{2^5\pi} \frac{e^{it^{1/2}(x_1-y_1)} e^{-(t-\lambda)^{1/2}(|x_2|+|y_2|)^{1/2}}}{t^{1/2}((t-\lambda)^{1/2})}.$$

Of course, the pointwise limits $\psi_\lambda = \lim_{\delta \rightarrow 0} \psi_\lambda^\delta$ cease to be square integrable; however, they still belong locally to L^2 , and in view of (3.1) they provide us with the generalized eigenfunction of $-\Delta_\Gamma$ in the form

$$\psi_\lambda = \omega_\lambda + R_{\Sigma, \nu}^{k_\alpha(\lambda)} (\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, \tag{3.4}$$

where $J_\Lambda \omega_\lambda$ is an embedding of ω_λ to $L^2(\nu_\Lambda)$. To find the S-matrix we have to investigate the behaviour of ψ_λ for $|x_1| \rightarrow \infty$. We employ the following result.

Lemma 3.1. *Let y belong to a compact $M \subset \mathbb{R}^2$ and $|x_1| \rightarrow \infty$, then*

$$G_\Sigma^{k_\alpha(\lambda)}(x-y) \approx s_\alpha(\lambda) e^{ik_\alpha(\lambda)|x_1-y_1|} e^{-\alpha/2(|x_2|+|y_2|)}.$$

⁴ To be fully specific, the formula is obtained from equation (4.8) of [EK2] after interchanging $x_1 \rightarrow x_1 - y_1$ and $a \rightarrow y_2$.

Proof. The argument is the same as in [EK2]. □

This allows us to formulate the sought conclusion.

Theorem 3.2. For a fixed $\lambda \in (-\frac{1}{4}\alpha^2, 0)$ the generalized eigenfunctions behave asymptotically as

$$\psi_\lambda(x) \approx \begin{cases} \mathcal{T}(\lambda) e^{ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} & \text{for } x_1 \rightarrow \infty \\ e^{ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} + \mathcal{R}(\lambda) e^{-ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} & \text{for } x_1 \rightarrow -\infty \end{cases} \quad (3.5)$$

where $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$ and $\mathcal{T}(\lambda), \mathcal{R}(\lambda)$ are the transmission and reflection amplitudes given respectively by

$$\mathcal{T}(\lambda) = 1 - s_\alpha(\lambda)((\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, J_\Lambda \omega_\lambda)_h$$

and

$$\mathcal{R}(\lambda) = s_\alpha(\lambda)((\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, J_\Lambda \bar{\omega}_\lambda)_h.$$

4. Concluding remarks

The general formulae for the S -matrix coefficients given in theorem 3.2 are not easy to handle and one would like to ask whether there are situations when there is a simple way, at least in a perturbative sense. Let us explain in more detail what such a result could look like, assuming that Γ is a C^4 smooth curve obtained by a local deformation of a straight line (conditions (a1), (a2) are then, of course, fulfilled) and α is large enough. We expect that \mathcal{T}, \mathcal{R} will be in this situation expressed in the leading order through the *local* geometry of Γ .

We may suppose without loss of generality that the curve is parametrized by its arc length being the graph of a function

$$\mathbb{R} \ni s \mapsto (\Gamma^{(1)}(s), \Gamma^{(2)}(s)) \in \mathbb{R}.$$

By assumption the curvature $\kappa(\cdot)$ of Γ is well defined and allows us to define a *comparison operator*, the same as in [EY1], by

$$K : D(K) \rightarrow L^2(\mathbb{R}), \quad K = -\frac{d^2}{ds^2} - \frac{1}{4}\kappa^2(s),$$

with the natural domain $D(K) := W^{2,2}(\mathbb{R})$. It is nothing but a one-dimensional Schrödinger operator with an attractive compactly supported C^2 smooth potential. The corresponding scattering problem is thus well posed and we denote by $\mathcal{T}_K(k), \mathcal{R}_K(k)$ the corresponding transmission and reflection amplitudes at a fixed momentum k . Denote by $\mathbf{S}_{\Gamma,\alpha}(\lambda)$ and $\mathbf{S}_K(\lambda)$ the on-shell S -matrices of $-\Delta_\Gamma$ and K at energy λ , respectively. Then we can make the following conjecture about the asymptotic behaviour.

Conjecture 4.1. For a fixed $k \neq 0$ and $\alpha \rightarrow \infty$ we have the relation

$$\mathbf{S}_{\Gamma,\alpha}(k^2 - \frac{1}{4}\alpha^2) \rightarrow \mathbf{S}_K(k^2). \quad (4.1)$$

The claim is inspired by the corresponding result about the discrete spectrum of such systems [EY1, EY2] which uses natural curvilinear coordinates in the vicinity of the curve to express the solution to the Schrödinger equation through that of the comparison problem plus an error term which vanishes as $\alpha \rightarrow \infty$. In the present case, however, one cannot use bracketing and minimax estimates and has to investigate instead directly the generalized eigenfunction in a

strip neighbourhood of Γ ; it is sufficient to find the behaviour of the solution in the straight asymptotic parts where $|x_1|$ is large. We postpone this analysis to a later publication.

Another open question which the considerations given in this paper raise is whether our results, notably theorem 3.2, extend to more general situations when Γ is no longer a local perturbation of a straight line but it remains asymptotically straight in a suitable sense. We expect that the answer will be positive; however, one will need to replace wave operators of section 2.4 by generalized ones referring to the asymptotes of Γ with an appropriate identification map.

Acknowledgments

The research has been partially supported by ASCR within the project K1010104 and by the Polish Ministry of Scientific Research and Information Technology under (solicited) grant no PBZ-Min-008/PO3/03.

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